

# On the $\alpha$ -Invariants of Cubic Surfaces with Eckardt Points

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## Abstract

In this paper, we show that the  $\alpha_{m,2}$ -invariant (introduced by Tian in [27] and [29]) of a smooth cubic surface with Eckardt points is strictly bigger than  $\frac{2}{3}$ . This can be used to simplify Tian's original proof of the existence of Kähler-Einstein metrics on such manifolds. We also sketch the computations on cubic surfaces with one ordinary double points, and outline the analytic difficulties to prove the existence of orbifold Kähler-Einstein metrics.

*Keywords:*  $\alpha_{m,2}$ -invariant; Kähler-Einstein metrics; cubic surface.

## 1 Introduction

A very important problem in complex geometry is the existence of canonical metrics, for example, the Kähler-Einstein metrics. An obvious necessary condition for the existence of a Kähler-Einstein metric is that the first Chern class of the manifold should be positive, zero or negative. Though the existence of Kähler-Einstein metrics was proved when  $c_1 \leq 0$  by Aubin (the  $c_1 < 0$  case) and Yau (both the  $c_1 < 0$  case and the  $c_1 = 0$  case) in the 1970's, the  $c_1 > 0$  case, i.e. the Fano case, is much more complicated and still not completely understood today. However, in complex dimension 2, the  $c_1 > 0$  case is completely solved by Tian in [26]. A complex surface with positive first Chern class is also called a Del Pezzo surface. By the classification of complex surfaces, the Del Pezzo surfaces are  $\mathbb{C}P^2$ ,  $\mathbb{C}P^1 \times \mathbb{C}P^1$ , and  $\mathbb{C}P^2$  blowing up at most eight points that are in general positions. According to [26], except for  $\mathbb{C}P^2$  blown up at one or two points, all the remaining Del Pezzo surfaces admit Kähler-Einstein metrics.

Up to now, the most effective way to prove the existence of Kähler-Einstein metrics is to use Tian's  $\alpha$ -invariants (and  $\alpha_G$ -invariants for a compact group  $G$ ) introduced in [24]. We now recall the definitions.

Let  $X$  be a compact Kähler manifold of dimension  $n$  with  $c_1(X) > 0$ .  $g$  is a Kähler metric with  $\omega_g := \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}} dz_i \wedge d\bar{z}_j \in c_1(X)$ . Define the space of Kähler potentials to be

$$P(X, g) = \{\varphi \in C^2(X; \mathbb{R}) \mid \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi > 0, \sup_X \varphi = 0\}.$$

We also define

$$\begin{aligned} P_m(X, g) &= \{\varphi \in P(X, g) \mid \exists \text{ a basis } s_0, \dots, s_{N_m} \text{ of } H^0(X, -mK_X), \\ &\quad \text{s.t. } \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi = \frac{\sqrt{-1}}{2m\pi} \partial\bar{\partial} \log(|s_0|^2 + \dots + |s_{N_m}|^2)\}. \end{aligned}$$

Here the functions  $|s_i|^2$ ,  $i = 0, \dots, N_m$  are only defined locally by choosing a local trivialization of  $K_X^{-m}$ . But it's easy to see that the (1,1)-form on the right hand side is independent of the trivialization we choose and is globally defined.

**Definition 1.1** (Tian [24], [25]). *The  $\alpha$ -invariant and  $\alpha_m$ -invariant of  $X$  are defined to be:*

$$\begin{aligned}\alpha(X) &:= \sup\{\alpha > 0 \mid \exists C_\alpha > 0, \text{ s.t. } \int_X e^{-\alpha\varphi} dV_g \leq C_\alpha, \forall \varphi \in P(X, g)\} \\ \alpha_m(X) &:= \sup\{\alpha > 0 \mid \exists C_\alpha > 0, \text{ s.t. } \int_X e^{-\alpha\varphi} dV_g \leq C_\alpha, \forall \varphi \in P_m(X, g)\}.\end{aligned}$$

If  $G$  is a compact subgroup of  $Aut(X)$ , choose a  $G$ -invariant Kähler form  $\omega_g$ , then we can define  $P_G(X, g)$  and  $P_{G,m}(X, g)$  by requiring the potentials to be  $G$ -invariant. Following the same procedure, we can also define the  $\alpha_G$ -invariant and  $\alpha_{G,m}$ -invariant.

We have the following criteria of Tian:<sup>1</sup>

**Theorem 1.1** (Tian [24]). *If  $\alpha_G(X) > \frac{n}{n+1}$  where  $n$  is the complex dimension of the Fano manifold  $X$  and  $G$  is a compact subgroup of  $Aut(X)$ , then  $X$  admits a  $G$ -invariant Kähler-Einstein metric.*

Recently, I. Cheltsov [3] computed all the  $\alpha$ -invariants and some of the  $\alpha_G$ -invariants for Del Pezzo surfaces. Combined with Tian and Tian-Yau's earlier work, this gives an alternative proof for the existence of Kähler-Einstein metrics on all the Del Pezzo surfaces of degree less than 7 except cubic surfaces with Eckardt points. Cheltsov showed that for cubic surfaces with Eckardt points, the  $\alpha$ -invariants are exactly  $\frac{2}{3}$ . One may ask whether we have  $\alpha_G > \frac{2}{3}$  for some nontrivial group  $G$  in the latter case. This is true for some special cubic surfaces with Eckardt points, for example the Fermat hypersurface in  $\mathbb{C}P^3$ . However, this is false in general.

**Example 1.1.** *Let  $X$  be the cubic surface defined by the equation*

$$z_1^3 + z_2^3 + z_3^3 + 6z_1z_2z_3 + z_0^2(z_1 + 2z_2 + 3z_3) = 0$$

where  $[z_0, z_1, z_2, z_3]$  are the homogeneous coordinates in  $\mathbb{C}P^3$ . Then according to [10] Table 10.3,  $Aut(X) = \mathbb{Z}_2$ , and  $X$  has exactly one Eckardt point, namely  $[1, 0, 0, 0]$ . It's easy to see that the anti-canonical divisor cut out by  $z_1 + 2z_2 + 3z_3 = 0$  consists of the three coplanar lines and is  $\mathbb{Z}_2$  invariant. By the equivariant version of Theorem 2.2 (See [7] for a proof.), we have  $\alpha_{\mathbb{Z}_2}(X) = \frac{2}{3}$ .

So for all cubic surfaces with Eckardt points, the only known proof for the existence of Kähler-Einstein metrics is still Tian's original one in [26]. The key idea of Tian's proof in [26] is to use his "partial  $C^0$ -estimate". There are two versions of "partial  $C^0$ -estimate". The weaker one (Theorem 5.1 of [26]) states that the function

$$\psi_m = \frac{1}{m} \log(|s_0|_{\omega_{KE}}^2 + \dots + |s_{N_m}|_{\omega_{KE}}^2)$$

for any smooth Kähler-Einstein cubic surface  $(X, \omega_{KE})$  satisfying  $Ric(\omega_{KE}) = \omega_{KE}$  has a uniform lower bound for some  $m$ , where  $\{s_i\}_{i=0}^{N_m}$  is an orthonormal basis of  $H^0(X, -mK_X)$  with

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<sup>1</sup>Another very convenient tool is the "multiplier ideal sheaves" introduced by Nadel in [17] and simplified by Demailly and Kollar in [8]. It's easy to see that their results are equivalent to Tian's theorem, see [22].

respect to the inner product induced by  $\omega_{KE}$ . The stronger one (Theorem 2.2 of [26]) says that this holds for any sufficiently large  $m$  satisfying  $m \equiv 0 \pmod{6}$ <sup>2</sup>. If we define the  $\alpha_{m,2}$ -invariant as follows:

**Definition 1.2** (Tian [27], [29]). *Let  $(X, \omega_g)$  be as above. The  $\alpha_{m,2}$ -invariant of  $X$  is defined to be:*

$$\begin{aligned} \alpha_{m,2}(X) := & \sup\{\alpha > 0 \mid \exists C_\alpha > 0, \text{ s.t. } \int_X (|s_1|_g^2 + |s_2|_g^2)^{-\frac{\alpha}{m}} dV_g \leq C_\alpha, \text{ for any} \\ & s_1, s_2 \in H^0(X, -mK_X) \text{ with } \langle s_i, s_j \rangle_g = \delta_{ij}\}. \end{aligned}$$

Then Tian proved the following criterion:

**Theorem 1.2** (Tian [26], also[27], [29]). *Let  $X$  be a smooth Del Pezzo surface obtained by blowing up  $\mathbb{C}P^2$  at 5 to 8 points in general position. If for some integer  $m \geq 0$ ,  $\psi_m$  has a uniform lower bound on the deformations of  $X$  that have Kähler-Einstein metrics, and*

$$\frac{1}{\alpha_m(X)} + \frac{1}{\alpha_{m,2}(X)} > 3,$$

*then  $X$  admits a Kähler-Einstein metric with positive scalar curvature.*

In the appendix of [26], Tian proved that  $\alpha_{6k,2} > \frac{2}{3}$ . Combining this with the stronger version of “partial  $C^0$ -estimate”, he proved the existence of Kähler-Einstein metrics on such manifolds using the above theorem.

In this paper, we prove the following theorem:

**Theorem 1.3** (Main Theorem). *Let  $X$  be a smooth cubic surface with Eckardt points, then for any integer  $m > 0$ , we have  $\alpha_{m,2}(X) > \frac{2}{3}$ .*

One application of this theorem is to give a simplified proof of Tian’s theorem in [26]. With our theorem in hand, the weaker version of “partial  $C^0$ -estimate” is sufficient to prove the existence of Kähler-Einstein metrics. We refer the reader to [27] and [29] for more details.

The proof of the main theorem will be given in section 4. In section 2, we will discuss basic properties of Tian’s invariants. Then we compute the  $\alpha$ -invariant for cubic surfaces with Eckardt points in section 3. This has already been done by Chelsov in [3]. We include a direct proof here for the reader’s convenience. In section 5, we sketch the computations on cubic surfaces with one ordinary double points since it is quite similar to the smooth case. Then we outline an approach to establish the existence of Kähler-Einstein metrics on singular stable cubic surfaces, and discuss briefly the extra analytic difficulties in this approach. The details will be presented elsewhere. We also include an appendix on relations between  $\alpha_m$ -invariants and the  $\alpha$ -invariant. This appendix is basically taken from [22].

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<sup>2</sup>This partial  $C^0$ -estimate should hold for any sufficiently large  $m$ . See Conjecture 6.4 of [30].

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## 2 Preliminaries on Tian's invariants

From the definitions of  $\alpha$ ,  $\alpha_m$  and  $\alpha_{m,2}$ , we see that these invariants are not so easy to compute. In particular, the uniform integration estimates involved in the definitions are difficult to verify. Fortunately, we have the following semi-continuity theorem for complex singularity exponents ( $= \log$  canonical thresholds):

**Theorem 2.1** (Demainly-Kollar [8], also see Phong-Sturm [21]). *Let  $X$  be a complex manifold. Let  $P(X)$  be the set of locally  $L^1$  plurisubharmonic functions on  $X$  equipped with the topology of  $L^1$  convergence on compact subsets. Let  $K$  be a compact subset of  $X$  and define the complex singularity exponent by*

$$c_K(\varphi) = \sup\{c > 0 \mid e^{-2c\varphi} \in L^1_{loc}(U) \text{ for some neighborhood } U \text{ of } K\}.$$

If  $\psi_i$  converges to  $\varphi$  in  $P(X)$  and  $c < c_K(\varphi)$ , then  $e^{-2c\psi_i}$  converges to  $e^{-2c\varphi}$  in  $L^1(U)$  for some neighborhood  $U$  of  $K$ .

The following proposition gives an alternative and easier way of computing  $\alpha_m(X)$ :

**Proposition 2.1.** *For any integer  $m > 0$ , we have*

$$\alpha_m(X) = \sup\{c > 0 \mid \int_X |s|_{h^m}^{-\frac{2c}{m}} dV_g < +\infty, \forall s \in H^0(X, -mK_X), s \neq 0\}.$$

**Remark 2.1.** *If we define  $c(s)$  to be the global complex singularity exponent of  $s$  (that is,  $c(s)$  is the supremum of the set of positive numbers  $c$  such that  $|s|^{-2c}$  is globally integrable), then the result of this proposition can be written as*

$$\alpha_m(X) = \inf\{m \cdot c(s) \mid s \in H^0(X, -mK_X), s \neq 0\}.$$

By Theorem 2.1, we can actually find a holomorphic section  $s \in H^0(X, -mK_X)$  satisfying  $\alpha_m(X) = m \cdot c(s)$ .

**Proof of Proposition 2.1:** We need only to show that

$$\begin{aligned} \alpha_m(X) &= \sup\{\alpha > 0 \mid \exists C_\alpha > 0, \text{ s.t. } \int_X |s|_{h^m}^{-\frac{2\alpha}{m}} dV_g \leq C_\alpha, \\ &\quad \forall s \in H^0(X, -mK_X) \text{ with } \int_X |s|_{h^m}^2 dV_g = 1\}. \end{aligned} \tag{\clubsuit}$$

For if this is true, the proposition follows easily from Theorem 2.1.

We now follow Tian's original computations([25], [26]).

Assume the hermitian metric  $h$  on  $K^{-1}$  satisfies  $Ric(h) = \omega_g$ . Fix an orthonormal basis  $s_0, \dots, s_{N_m}$  of  $H^0(X, -mK_X)$  with respect to  $h$  and  $\omega_g$ . For any  $\varphi \in P_m(X, g)$ , there exists a basis of  $H^0(X, -mK_X)$   $s'_0, \dots, s'_{N_m}$ , such that

$$\begin{aligned}\omega_g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi &= \frac{\sqrt{-1}}{2m\pi} \partial \bar{\partial} \log(|s'_0|^2 + \dots + |s'_{N_m}|^2) \\ &= \frac{\sqrt{-1}}{2m\pi} \partial \bar{\partial} \log(|s'_0|_{h^m}^2 + \dots + |s'_{N_m}|_{h^m}^2) + \omega_g\end{aligned}$$

Set

$$\tilde{\varphi} := \frac{1}{m} \log(|s'_0|_{h^m}^2 + \dots + |s'_{N_m}|_{h^m}^2),$$

then  $\varphi = \tilde{\varphi} - \sup_X \tilde{\varphi}$ . Since the value of  $\tilde{\varphi}$  doesn't change under unitary transformations on  $H^0(X, -mK_X)$  with respect to  $h$  and  $\omega_g$ , we may assume further that  $s'_i = \lambda_i s_i$ ,  $0 < \lambda_0 \leq \dots \leq \lambda_{N_m}$ . So we can write  $\tilde{\varphi}$  as

$$\tilde{\varphi} = \frac{1}{m} \log(\lambda_0^2 |s_0|_{h^m}^2 + \dots + \lambda_{N_m}^2 |s_{N_m}|_{h^m}^2).$$

Observe that

$$\sup_X \tilde{\varphi} = \frac{1}{m} \log(\lambda_{N_m}^2) + O(1),$$

we can write

$$\varphi = \frac{1}{m} \log\left(\sum_{i=0}^{N_m} \lambda_i^2 |s_i|_{h^m}^2\right) + O(1)$$

with  $0 < \lambda_0 \leq \dots \leq \lambda_{N_m} = 1$ . Then the equality ( $\clubsuit$ ) follows easily from this expression and Theorem 2.1.  $\square$

**Theorem 2.2** (Demainly [7]). *For any Fano manifold  $X$ , we have*

$$\alpha(X) = \inf\{\alpha_m(X)\} = \lim_{m \rightarrow +\infty} \alpha_m(X).$$

A simple proof of Theorem 2.2 will be given in the appendix, which is basically taken from the author's thesis in preparation. Note that though Demainly's proof looks more complicated than the proof we give here, his proof can yield more information in the equivariant case. We refer the interested readers to his paper for more details.

**Remark 2.2.** *A conjecture of Tian claims that for any Fano manifold  $X$ , one has  $\alpha(X) = \alpha_m(X)$  when  $m$  is sufficiently large. We will discuss this problem in a separate paper.*

Now we state a similar proposition for  $\alpha_{m,2}$ -invariants, whose proof is quite easy and thus omitted.

**Proposition 2.2.** Let  $(X, \omega_g)$  be as in Definition 1.1. Then we have:

$$\begin{aligned} \alpha_{m,2}(X) &= \sup\{\alpha > 0 \mid \int_X (|s_1|_g^2 + |s_2|_g^2)^{-\frac{\alpha}{m}} dV_g < +\infty, \text{ for any} \\ &\quad s_1, s_2 \in H^0(X, -mK_X) \text{ with } \langle s_i, s_j \rangle_g = \delta_{ij}\}. \end{aligned}$$

### 3 The $\alpha$ invariants of cubic surfaces with Eckardt points

Let  $X$  be a smooth cubic surface in  $\mathbb{C}P^3$ . It's well known that there are exactly 27 lines on  $X$ . If we realize  $X$  as  $\mathbb{C}P^2$  blowing up 6 generic points  $p_1, \dots, p_6$ , then the 27 lines are:

- the exceptional divisors:  $E_1, \dots, E_6$ ;
- the strict transforms of lines passing through 2 of the 6 points:  $L_{12}, \dots, L_{56}$ ;
- the strict transforms of the quadrics that avoids only 1 of the 6 points:  $F_1, \dots, F_6$ .

It's easy to check that each line above intersects with other 10 lines, and that if 2 lines intersect, then there is a unique other line that intersects them both. If it happens that there are three coplanar lines intersecting at one point  $p$  on  $X$ , then we call  $p$  an “Eckardt point”. Note that a generic cubic surface does not have any Eckardt points. For detailed information about cubic surfaces, we refer the reader to the books [11], [12] and [10].

We shall prove the following theorem of Cheltsov in this section.

**Theorem 3.1** (Cheltsov[3]). *Let  $X$  be a smooth cubic surface with Eckardt points, then for any integer  $m > 0$ ,  $\alpha_m(X) = \alpha_1(X) = \frac{2}{3}$ . In particular, by Theorem 2.2,  $\alpha(X) = \frac{2}{3}$ .*

#### 3.1 Computing $\alpha_1(X)$

To compute the  $\alpha_1$ -invariant of our cubic surface, by Proposition 2.1, we need only to consider the singularities cut out by anti-canonical sections. This is done, for example, in [26] and [19]. The most “singular” sections are exactly those defined by triples of lines intersecting at Eckardt points. It's easy to see that the singularity exponents of these sections are equal to  $\frac{2}{3}$ .

#### 3.2 Computing $\alpha(X)$

Now we show that  $\alpha(X) = \alpha_1(X)$ . The main tool is the following theorem:<sup>3</sup>

**Theorem 3.2** (Nadel vanishing theorem). *Let  $X$  be a smooth complex projective variety, let  $D$  be any  $\mathbb{Q}$ -divisor on  $X$ , and let  $L$  be any integral divisor such that  $L - D$  is nef and big. Denote the multiplier ideal sheaf of  $D$  by  $\mathcal{J}(D)$ , then*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)) = 0$$

for any  $i > 0$ .

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<sup>3</sup>Actually, what we use in this paper is just the connectedness of “multiplier ideal subschemes”, which in fact can also be proved directly by Hörmander's  $L^2$  method, see [27].

**Proof of Theorem 3.1:** Suppose  $\alpha(X) < \alpha_1(X)$ , then there is an integer  $m$  such that  $\alpha_m(X) < \alpha_1(X)$ . Then by definition, there is a nontrivial holomorphic section  $s$  of  $K_X^{-m}$  such that  $c(s) < \frac{\lambda}{m}$ , where  $\lambda \in \mathbb{Q}$  and  $\lambda < \frac{2}{3}$ . We denote the corresponding effective divisor by  $Z(s) \in |-mK_X|$ .

Now we need a lemma:

**Lemma 3.1.** *For any  $0 < \lambda \leq \frac{2}{3}$  and any nonzero holomorphic section  $s \in H^0(X, -mK_X)$ , the locus of non-integrable points of  $|s|^{-\frac{2\lambda}{m}}$  is a single point.*

**Proof of Lemma 3.1:** Denote by  $Z(s)$  the effective divisor defined by the section  $s$ . Apply Nadel's theorem to the sheaf  $\mathcal{J}(\frac{\lambda}{m}Z(s))$  and the integral divisor  $-K_X$ , we know that the locus of non-integrable points of  $|s|^{-\frac{2\lambda}{m}}$  should be a connected subset of  $X$ . We denote the locus by  $C$ . So we need only to show that  $C$  does not contain one dimensional parts.

Suppose this is not true. Write  $C = \cup_i C_i$ , where the  $C_i$ 's are different irreducible curves. Then we can write  $Z(s)$  as

$$Z(s) = \sum_i \mu_i C_i + \Omega$$

where  $\Omega$  is an effective divisor whose support doesn't contain any of the  $C_i$ 's and the  $\mu_i$ 's are integers such that  $\frac{\lambda \mu_i}{m} \geq 1$ .

First, we know that every  $C_i$  is a smooth rational curve, for if not, we have  $C_i^2 > 0$ , then

$$0 \leq 2p_a(C_i) - 2 = (C_i + K_X) \cdot C_i \leq (1 - \frac{\mu_i}{m})C_i^2 - \sum_{j \neq i} \frac{\mu_j}{m} C_j \cdot C_i < 0$$

A contradiction.

On the other hand, we have

$$3m = -Z(s) \cdot K_X = -\sum_i \mu_i C_i \cdot K_X - K_X \cdot \Omega \geq \frac{3m}{2} \sum_i C_i \cdot (-K_X)$$

So there are three possibilities:

1.  $\Omega$  is empty and there are two  $C_i$ 's, both among the 27 lines;
2.  $\Omega$  is empty and there is only one  $C_i$ , with  $C_i^2 = 0$  and  $K_X \cdot C_i = -2$ ;
3. There is only one  $C_i$ , and it is one of the 27 lines.

In case 1,  $\lambda = \frac{2}{3}$ ,  $\mu_1 = \mu_2 = \frac{3m}{2}$ , and  $Z(s) = \frac{3m}{2}(C_1 + C_2)$ . But  $(C_1 + C_2)$  can not be an ample divisor. A contradiction.

In case 2,  $\lambda = \frac{2}{3}$ ,  $\mu = \frac{3m}{2}$ , and  $Z(s) = \frac{3m}{2}C_1$ . Since  $C_1^2 = 0$ ,  $C_1$  can not be ample, a contradiction.

In case 3, write  $Z(s) = \mu C_1 + \Omega$ . Choose a birational morphism  $\pi$  from  $X$  to  $\mathbb{C}P^2$  such that  $\deg\pi(C_1) = 2$ . Then

$$3\lambda = \pi^*H \cdot \frac{\lambda}{m}Z(s) \geq \pi^*H \cdot \frac{\lambda}{m}\mu C_1 = H \cdot \frac{\lambda}{m}\mu\pi(C_1) = \frac{2\lambda}{m}\mu \geq 2$$

If  $\lambda < \frac{2}{3}$ , this is already a contradiction. If  $\lambda = \frac{2}{3}$ , then  $\mu = \frac{3m}{2}$  and  $\Omega$  consists of exceptional divisors, i.e. lines. Write

$$Z(s) = \frac{3m}{2}C_1 + \sum_i \kappa_i L_i.$$

If  $L$  is a line intersects with  $C_1$ , then  $L$  must be contained in  $\Omega$ , for otherwise  $m = L \cdot Z(s) \geq \frac{3m}{2}$ . A contradiction. So  $\Omega$  must contain the 10 lines having positive intersection numbers with  $C_1$ . On the other hand, if  $L_i \cdot C_1 = 1$ , then

$$m = L_i \cdot Z(s) \geq \frac{3m}{2} - \kappa_i,$$

so  $\kappa_i \geq \frac{m}{2}$ . However,

$$m = C_1 \cdot Z(s) = -\frac{3m}{2} + \sum_i \kappa_i L_i \cdot C_1$$

So there are at most 5  $L_i$ 's having positive intersection numbers with  $C_1$ . A contradiction.  $\square$

**Remark 3.1.** *The above lemma actually holds for any  $0 < \lambda < 1$ . We refer the reader to [18] for a proof.*

Now we continue to prove Theorem 3.1. Denote the point in the above lemma by  $p$ . Now choose a birational morphism  $\pi$  from  $X$  to  $\mathbb{C}P^2$  such that it is an isomorphism near  $p$ . Then  $\pi(Z(s))$  is an effective divisor of  $\mathbb{C}P^2$ . It's obvious that  $\pi(Z(s)) \in | -mK_{\mathbb{C}P^2}|$ . Choose a generic line  $L$  of  $\mathbb{C}P^2$  that doesn't pass  $\pi(p)$ . Let's now consider the  $\mathbb{Q}$ -divisor  $\Omega := \frac{\lambda}{m}\pi(Z(s)) + L$  which is numerically equivalent to  $(3\lambda + 1)H$ . Consider the multiplier ideal sheaf  $\mathcal{J}(\Omega)$ . By Nadel's vanishing theorem, the multiplier ideal subscheme associated with  $\mathcal{J}(\Omega)$  should be connected. But from our construction, its support should be  $\{\pi(p)\} \cup L$ , which is obviously not connected. A contradiction.  $\square$

## 4 Proof of the main theorem

The key to the proof of the main theorem is the following:

**Theorem 4.1.** *Suppose  $X$  is a smooth cubic surface in  $\mathbb{C}P^3$  with Eckardt points,  $m$  is a positive integer, then if  $s \in H^0(X, -mK_X)$  is a section such that  $c(s) = \frac{2}{3m}$ , then there exists a section  $s_1 \in H^0(X, -K_X)$  such that  $s = s_1^{\otimes m}$ . Moreover, the support of  $Z(s_1)$  consists of three lines intersecting at one of the Eckardt points.*

Actually, Tian proved the theorem in the case of  $m = 6$  in the appendix of [26]. Our proof is greatly inspired by his. The proof is based on the following observations:

**Lemma 4.1.** *Suppose  $s = s_1 \otimes s_2$  with  $s_1 \in H^0(X, -K_X)$ ,  $s_2 \in H^0(X, -(m-1)K_X)$  and  $c(s) = \frac{2}{3m}$ , then  $c(s_1) = \frac{2}{3}$  and  $c(s_2) = \frac{2}{3(m-1)}$ .*

**Proof:** We have  $c(s_1) \geq \frac{2}{3}$  and  $c(s_2) \geq \frac{2}{3(m-1)}$  by Theorem 3.1. Then the lemma is trivial by the Hölder inequality.  $\square$

**Lemma 4.2.** *If  $s \in H^0(X, -mK_X)$  is a holomorphic section and  $\text{mult}_p s > m$  for some point  $p$ . If  $p$  lies on one of the lines  $L$ , then  $L$  is contained in the support of  $Z(s)$ . In particular, if  $\text{mult}_p s > m$ , and  $p$  is an Eckardt point, then the three lines through  $p$  are all contained in the support of  $Z(s)$ , hence  $s = s_1 \otimes s_2$  with  $s_1 \in H^0(X, -K_X)$ ,  $s_2 \in H^0(X, -(m-1)K_X)$ .*

**Proof:** Suppose  $L$  is not contained in the support of  $Z(s)$ , then

$$m = L \cdot Z(s) \geq \text{mult}_p s > m$$

A contradiction.  $\square$

**Lemma 4.3.** *If  $f \in \mathcal{O}_{\mathbb{C}^2,0}$  with  $\text{mult}_0 f = k$ , then  $\frac{1}{k} \leq c_0(f) \leq \frac{2}{k}$ , and if moreover  $c_0(f) = \frac{1}{k}$ , then locally  $f = gh^k$ , where  $g(0) \neq 0$  and  $\text{mult}_0 h = 1$ .*

**Proof:** This lemma follows easily from the fact that in dimension two, one can compute the singularity exponent via Newton polygons for some analytic coordinates. We refer the reader to Varchenko's paper [33], the appendix of Tian's paper [26] and the book of Kollar, Smith and Corti [15] for detailed proofs.  $\square$

Based on these lemmas, we need only to show that the only point  $p$  where  $c_p(s) = \frac{2}{3m}$  is an Eckardt point. Actually the arguments in [3] already imply this, but his proof is more complicated and uses some properties of Geiser involutions. So we give a simple proof here, which avoids Geiser involutions but still uses some observations of Cheltsov [3] and Tian [26].<sup>4</sup>

**Proof of Theorem 4.1:** Suppose  $p$  is not an Eckardt point, then there are three possibilities:

1.  $p$  doesn't belong to any of the lines;
2.  $p$  belongs to exactly one line;
3.  $p$  belongs to exactly two lines.

We shall rule them out one by one. First, note that by Lemma 3.1 and Lemma 4.3, we have  $\text{mult}_p s > \frac{3m}{2}$ .

### Case 1

In this case, we may choose a  $D \in |-K_X|$  which has multiplicity 2 at  $p$ . Then if  $D$  is not contained in the support of  $Z(s)$ , we have

$$3m = D \cdot Z(s) \geq 2\text{mult}_p s > 3m.$$

A contradiction.

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<sup>4</sup>Theorem 4.1 here is a special case of Theorem 4.1 of Cheltsov [2], which was not known to me when I wrote the first version of this paper. Here I give an alternative and more direct proof. The readers can consult [2] for the proof of a more general result.

### Case 2

In this case, there is a section  $s' \in H^0(X, -K_X)$  such that  $Z(s') = L_1 + D$  where  $L_1$  is a line and  $D$  is a quadratic curve intersecting with  $L_1$  at  $p$ . Then  $L_1$  is contained in the support of  $Z(s)$ , so  $D$  is not contained in the support of  $Z(s)$  in view of Lemma 4.1. Thus

$$2m = D \cdot Z(s) \geq \text{mult}_p D \cdot \text{mult}_p s > \frac{3m}{2} \cdot \text{mult}_p D,$$

which implies  $\text{mult}_p D = 1$ .

Write  $Z(s) = \mu L_1 + \Omega$ , where  $\Omega$  is an effective divisor whose support doesn't contain  $L_1$ . We have

$$m = L_1 \cdot Z(s) \geq -\mu + \text{mult}_p \Omega = \text{mult}_p s - 2\mu$$

and

$$2m = D \cdot Z(s) \geq 2\mu + \text{mult}_p \Omega = \text{mult}_p s + \mu,$$

which imply  $\text{mult}_p s \leq \frac{5m}{3}$  and  $\frac{m}{4} < \mu < \frac{m}{2}$ .

We blow up  $X$  at  $p$  to get a surface  $U$ ,  $\pi : U \rightarrow X$ . For any divisor  $F$  of  $X$ , denote by  $\bar{F}$  the strict transform of  $F$ . We have

$$\pi^*(K_X + \frac{2}{3m}Z(s)) = K_U + \frac{2}{3m}(\mu \bar{L}_1 + \bar{\Omega}) + (\frac{2}{3m}\text{mult}_p s - 1)E.$$

Since the pair  $(X, \frac{2}{3m}Z(s))$  is not log terminal at  $p$ , there is a point  $Q \in E$  satisfying

$$\frac{2}{3m}(\mu \text{mult}_Q \bar{L}_1 + \text{mult}_Q \bar{\Omega}) + (\frac{2}{3m}\text{mult}_p s - 1) \geq 1,$$

that is

$$\mu \text{mult}_Q \bar{L}_1 + \text{mult}_Q \bar{\Omega} + \text{mult}_p s \geq 3m.$$

Now we prove that  $Q \notin \bar{L}_1$ . If  $Q \in \bar{L}_1$ , then since  $\frac{2\mu}{3m} < 1$ , by the Fubini theorem and Theorem 2.1, we know that

$$\bar{L}_1 \cdot [\frac{2}{3m}\bar{\Omega} + (\frac{2}{3m}\text{mult}_p s - 1)E] \geq 1,$$

which implies that  $\mu > m$ . A contradiction.

So we have

$$\text{mult}_Q \bar{\Omega} + \text{mult}_p s \geq 3m.$$

Then we can see that  $\text{mult}_Q \bar{\Omega} \geq \frac{4m}{3}$  and hence  $\text{mult}_p \Omega \geq \frac{4m}{3}$ . Moreover, it's easy to see that actually  $\text{mult}_p \Omega = \frac{4m}{3}$  and  $\mu = \frac{m}{3}$ .

If  $m$  is not a multiple of 3, this already leads to a contradiction. If  $m = 3k$ , then  $Z(s) = kL_1 + \Omega$  with  $\text{mult}_p \Omega = 4k$ . In this case,  $L_1$  intersects with  $\Omega$  only at  $p$ , and  $L_1$  is not tangent to  $\Omega$  at  $p$ . By Theorem 2.1 and the Fubini theorem, the singularity exponent of  $s$  at  $p$  is at least  $\frac{1}{4k}$  which is bigger than  $\frac{2}{3m} = \frac{2}{9k}$ . A contradiction.

### Case 3

In this case, there is a section  $s' \in H^0(X, -K_X)$ , with  $Z(s') = L_1 + L_2 + L_3$  where  $L_1$  and  $L_2$  intersects at  $p$  and  $L_3$  is the other line coplanar with  $L_1, L_2$  and  $p \notin L_3$ . Firstly  $L_1$  and  $L_2$  must be contained in the support of  $Z(s)$  as before.

By Lemma 4.1,  $L_3$  is not contained in the support of  $Z(s)$ . Write  $Z(s) = \mu L_1 + \nu L_2 + D$ , then

$$m = L_1 \cdot Z(s) = -\mu + \nu + L_1 \cdot D \geq -\mu + \nu + \text{mult}_p D$$

$$m = L_2 \cdot Z(s) = \mu - \nu + L_2 \cdot D \geq \mu - \nu + \text{mult}_p D$$

$$m = L_3 \cdot Z(s) = \mu + \nu + L_3 \cdot D \geq \mu + \nu$$

These imply that

$$\mu + \nu \leq m, \quad \frac{m}{2} < \text{mult}_p D \leq m \quad \text{and} \quad \mu > \frac{m}{4}, \quad \nu > \frac{m}{4}.$$

As in Case 2, we blow up  $X$  at  $p$  to obtain a surface  $U$ . We have

$$\pi^*(K_X + \frac{2}{3m}Z(s)) = K_U + \frac{2}{3m}(\mu \bar{L}_1 + \nu \bar{L}_2 + \bar{D}) + (\frac{2}{3m}\text{mult}_p s - 1)E.$$

As before, there is a point  $Q$  on  $E$  satisfying

$$\mu \text{mult}_Q \bar{L}_1 + \nu \text{mult}_Q \bar{L}_2 + \text{mult}_Q \bar{D} + \text{mult}_p s \geq 3m.$$

It's easy to see that  $Q \notin \bar{L}_1 \cup \bar{L}_2$ , so the above inequality reduces to (with  $\mu + \nu \leq m$  in mind)

$$\text{mult}_Q \bar{D} + \text{mult}_p D \geq 2m.$$

Since  $\text{mult}_p D \leq m$ , we must have  $\mu = \nu = \frac{m}{2}$ , and  $\text{mult}_p D = m$ .

If  $m$  is odd, this already leads to a contradiction. Now suppose  $m = 2k$ . We can write the section  $s$  locally as  $s = z_1^k z_2^k h$ , with  $\text{mult}_0 h = 2k$ . By the Hölder inequality and the fact that

$$c_0(s) = \frac{1}{3k}, \quad c_0(z_1^k z_2^k) = \frac{1}{k},$$

we know that  $c_0(h) \leq \frac{1}{2k}$ . So by Lemma 4.3, we can write  $\Omega$  locally at  $p$  as  $\Omega = 2kC$ , where  $C$  is a curve regular at  $p$  and not tangent to  $L_1$  or  $L_2$ . Then by blowing up  $p$  we get a log resolution for the pair  $(X, Z(s))$  near  $p$ . It's easy to see that

$$c_p(s) = \text{lct}_p(X, Z(s)) = \frac{1}{2k} > \frac{1}{3k} = \frac{2}{3m}.$$

A contradiction.  $\square$

Now let's turn to the proof of the main theorem:

**Proof of Theorem 1.2:** Obviously,  $\alpha_{m,2}(X) \geq \alpha_m(X) = \frac{2}{3}$ . It's easy to show that there are orthogonal sections  $s_1, s_2 \in H^0(X, -mK_X)$  such that

$$\alpha_{m,2}(X) = mc((|s_1|_g^2 + |s_2|_g^2)^{\frac{1}{2}}).$$

To prove the theorem, by compactness arguments, it suffices to show that at every point  $p \in X$ ,

$$c_p((|s_1|_g^2 + |s_2|_g^2)^{\frac{1}{2}}) > \frac{2}{3m}.$$

This is simple. Suppose

$$c_p((|s_1|_g^2 + |s_2|_g^2)^{\frac{1}{2}}) = \frac{2}{3m}$$

for some point  $p$ . By comparing  $(|s_1|_g^2 + |s_2|_g^2)^{\frac{1}{2}}$  with  $|s_1|_g$  and  $|s_2|_g$  respectively, we know that  $c_p(s_i) = \frac{2}{3m}$ . Hence by Theorem 4.1,  $p$  must be an Eckardt point and  $s_1 = \lambda s_2$  for some constant  $\lambda$ . A contradiction.  $\square$

## 5 Cubic surface with one ordinary double point

Now we assume that the cubic surface  $X$  has one ordinary double point  $O$ . If we blow up  $O$ , then we will get the minimal resolution of  $X$ . Denote the blow up map by  $\pi : \tilde{X} \rightarrow X$ , then we have  $K_{\tilde{X}} = \pi^*K_X$ . So the  $\alpha_m$  and  $\alpha_{m,2}$  invariants of  $X$  equal that of  $\tilde{X}$ . In this section, we estimate these invariants. Since the computation on  $\tilde{X}$  is quite similar to that of the smooth case, we shall be sketchy here.

### 5.1 The $\alpha_m$ invariants of $\tilde{X}$

It is well known that  $\tilde{X}$  can be realized as  $\mathbb{C}P^2$  blown up at six “almost general” points  $p_1, \dots, p_6$ . Here “almost general” means that three of the six points lie on a common line, but no four of them lie on a common line and these six points are not on a quadratic curve. ([11], [10]) Suppose  $p_1, p_2, p_3$  lie on a common line whose strict transform is the  $(-2)$ -curve  $C$ . We denote the exceptional divisors by  $E_1, \dots, E_6$ ; denote the strict transforms of the line passing through  $p_i$  and  $p_j$  by  $L_{ij}$ ; and denote the strict transform of the quadratic curve passing through  $p_1, \dots, \hat{p}_i, \dots, p_6$  by  $F_i$ . It is easy to see that there are 21  $(-1)$ -curves on  $\tilde{X}$ :

- $E_1, E_2, E_3, E_4, E_5, E_6$ ;
- $L_{14}, L_{24}, L_{34}, L_{15}, L_{25}, L_{35}, L_{16}, L_{26}, L_{36}, L_{45}, L_{46}, L_{56}$ ;
- $F_1, F_2, F_3$ .

There are six  $(-1)$ -curves that intersect with the  $(-2)$ -curve  $C$ :  $E_1, E_2, E_3$  and  $L_{45}, L_{46}, L_{56}$ . For any of these six curves, there is a smooth rational curve passing through the intersection point of the  $(-1)$ -curve and  $C$ , and together these three curves constitute an anticanonical divisor of  $\tilde{X}$ . This fact in particular implies that  $\alpha_m(\tilde{X}) \leq \frac{2}{3}$ . In [4], I. Cheltsov proved  $\alpha_m(\tilde{X}) = \frac{2}{3}$ . (See also [20] for the computation of  $\alpha_1$ .) Actually, this also follows easily from the following lemma:

**Lemma 5.1.** *For any  $0 < \lambda \leq \frac{2}{3}$  and any nonzero holomorphic section  $s \in H^0(X, -mK_{\tilde{X}})$ , the locus of non-integrable points of  $|s|^{-\frac{2\lambda}{m}}$  is connected. If it is not an isolated point, then it must be the  $(-2)$ -curve  $C$ , and in this case,  $\lambda = \frac{2}{3}$  and  $m$  must be even. Moreover, we have*

$$Z(s) = \frac{3m}{2}C + \frac{m}{2}(E_1 + E_2 + E_3 + L_{45} + L_{46} + L_{56}).$$

**Proof:** It suffices to consider the case when  $\lambda = \frac{2}{3}$ . As in the proof of Lemma 3.1, by Nadel's vanishing theorem, we know that the locus of non-integrable points of  $|s|^{-\frac{4}{3m}}$ , denoted by  $LT(s)$ , is a connected subset of  $\tilde{X}$ .

If  $LT(s)$  is not an isolated point, then we may assume that  $LT(s) = \cup_i^k C_i$ , where these  $C'_i$ s are different irreducible curves. We can also write

$$Z(s) = \sum_i^k \mu_i C_i + \Omega$$

where  $\Omega$  is an effective divisor whose support does not contain any of the  $C'_i$ s. By definition of  $C_i$ , we have  $\mu_i \geq \frac{3m}{2}$ . It's easy to see that each  $C_i$  is a smooth rational curve. Moreover, we have

$$3m = -Z(s) \cdot K_{\tilde{X}} \geq -\sum_i \mu_i C_i \cdot K_{\tilde{X}} \geq \frac{3m}{2} \sum_i C_i \cdot (-K_{\tilde{X}}) \Rightarrow \sum_i^k C_i \cdot (-K_{\tilde{X}}) \leq 2.$$

There are three possibilities:

1.  $\sum_i^k C_i \cdot (-K_{\tilde{X}}) = 0$ ;
2.  $\sum_i^k C_i \cdot (-K_{\tilde{X}}) = 1$ ;
3.  $\sum_i^k C_i \cdot (-K_{\tilde{X}}) = 2$ .

We now consider the three cases one by one.

In case 1, there can be only one irreducible curve in  $\sum_i C_i$ , and it must be the  $(-2)$ -curve  $C$ . So we can write  $Z(s) = \mu C + \Omega$ , with  $\mu \geq \frac{3m}{2}$ . Choose any  $(-1)$ -curve  $E$  that has positive intersection number with  $C$ . By computing the intersection number  $E \cdot Z(s)$ , it is easy to see that  $\Omega$  must contain  $E$  with multiplicity at least  $\frac{m}{2}$ . But if this is true, we will have

$$3m = \Omega \cdot (-K_{\tilde{X}}) \geq 6 \cdot \frac{m}{2} = 3m,$$

so we must have

$$\Omega = \frac{m}{2}(E_1 + E_2 + E_3 + L_{45} + L_{46} + L_{56})$$

and  $\mu = \frac{3m}{2}$ . So  $m$  must be even. In this case, it is easy to check that

$$\frac{3m}{2}C + \frac{m}{2}(E_1 + E_2 + E_3 + L_{45} + L_{46} + L_{56}) \in |-mK_{\tilde{X}}|,$$

and  $LT(s) = C$ .

In case 2, we have either  $LT(s) = C_1$  or  $LT(s) = C \cup C_1$ , where  $C_1$  is a  $(-1)$ -curve.

If  $LT(s) = C_1$ , we can write

$$Z(s) = \frac{3m}{2}C_1 + \Omega$$

where  $\Omega$  does not contain  $C_1$ . Then  $C_1 \cdot \Omega = \frac{5m}{2}$ . On the other hand,  $\Omega$  must contain any  $(-1)$ -curve that intersects with  $C_1$ , with coefficient at least  $\frac{m}{2}$ . So there are at most 5 such  $(-1)$ -curves. In this case, it is easy to see that  $C_1 \cdot C = 1$ . Thus  $C \cdot \Omega = -\frac{3m}{2}$ , which implies that  $\Omega$  contains  $C$  with multiplicity at least  $\frac{3m}{4}$ . But

$$\frac{5m}{2} = C_1 \cdot \Omega \geq 5 \cdot \frac{m}{2} + \frac{3m}{4} > \frac{5m}{2},$$

a contradiction.

If  $LT(s) = C \cup C_1$ , then we can write

$$Z(s) = \frac{3m}{2}(C + C_1) + \Omega,$$

where  $\Omega$  contains neither  $C$  nor  $C_1$ . Then  $-K_{\tilde{X}} \cdot \Omega = \frac{3m}{2}$ . However, as before, we can show that  $\Omega$  must contain at least 5  $(-1)$ -curves that have positive intersection numbers with  $C_1$ , with multiplicity greater than or equal to  $\frac{m}{2}$ . This implies that  $-K_{\tilde{X}} \cdot \Omega \geq \frac{5m}{2} > \frac{3m}{2}$ . A contradiction.

In case 3, we have either  $LT(s) = C_1 \cup C_2$  or  $LT(s) = C \cup C_1 \cup C_2$ , where  $C_1$  and  $C_2$  are both  $(-1)$ -curves.

If  $LT(s) = C_1 \cup C_2$ , then

$$Z(s) = \frac{3m}{2}(C_1 + C_2) + \Omega,$$

where  $\Omega$  contains neither  $C_1$  nor  $C_2$ . Since  $-K_{\tilde{X}} \cdot \Omega = 0$ , we have  $\Omega = kC$  for some nonnegative integer  $k$ . Choose any  $(-1)$ -curve  $L$  such that  $L \cap C = \emptyset$ , then

$$m = L \cdot Z(s) = \frac{3m}{2}(L \cdot C_1 + L \cdot C_2).$$

But this is impossible, since the right hand side never equals  $m$ .

If  $LT(s) = C \cup C_1 \cup C_2$ , then it is easy to see that actually  $Z(s) = \frac{3m}{2}(C + C_1 + C_2)$ . We can easily get a contradiction as above.  $\square$

Since the canonical bundle of  $\tilde{X}$  is nef and big, we can use Nadel's vanishing theorem as in the proof of Theorem 3.1 to get the following:

**Proposition 5.1.** *For any integer  $m > 0$ , we have  $\alpha_m(\tilde{X}) = \frac{2}{3}$ .*

## 5.2 The $\alpha_{m,2}$ invariants of $\tilde{X}$

We have the following theorem:

**Theorem 5.1.** *The  $\alpha_{m,2}$  invariants of  $\tilde{X}$  is strictly bigger than  $\frac{2}{3}$ .*

As in the proof of Theorem 1.2, the key to the proof of Theorem 5.1 is the following proposition:

**Proposition 5.2.** *If  $s \in H^0(\tilde{X}, -mK)$  satisfies  $c(s) = \frac{2}{3m}$ , and  $LT(s)$  is an isolated point, then  $s = s_1^{\otimes m}$  where  $s_1 \in H^0(\tilde{X}, -K)$  and  $c(s) = \frac{2}{3}$ .*

**Proof:** Suppose  $|s|^{-\frac{4}{3m}}$  is not integrable around a point  $p$ . Then in view of Lemma 4.3 and Lemma 5.1, we have

$$\text{mult}_p s > \frac{3m}{2}.$$

This in particular implies that  $p$  lies on  $C$  or some  $(-1)$ -curves, for otherwise we can find an effective divisor  $D \in |-K|$  with  $\text{mult}_p D = 2$ . By Lemma 4.1,  $D \not\subseteq \text{supp } Z(s)$ . So we have

$$3m = D \cdot Z(s) \geq 2\text{mult}_p s,$$

which leads to a contradiction.

Moreover, if  $p$  is not an Eckardt point, then as observed by Cheltsov (Theorem 3.2 of [5])  $p$  must lie on the  $(-2)$ -curve  $C$ . Actually, if  $p \notin C$ , then we can repeat the proof of Theorem 4.1 to show that  $p$  is an Eckardt point.

Now we prove that  $p$  also lies on a  $(-1)$ -curve. Suppose not, then it is easy to see that there is an irreducible curve  $D$  with  $\text{mult}_p D = 2$  and  $D + C \in |-K|$ . By Lemma 4.1,  $D \not\subseteq \text{supp } Z(s)$ . So we have

$$3m = D \cdot Z(s) \geq 2\text{mult}_p s > 3m.$$

A contradiction.

Denote the  $(-1)$ -curve through  $p$  by  $L$ , then there is a unique irreducible curve  $D$  such that  $p \in D$  and  $C + L + D \in |-K|$ . Write  $Z(s)$  as  $Z(s) = \mu C + \nu L + \Omega$ , where  $\Omega$  is an effective divisor whose support contains neither  $C$  nor  $L$ . Assume  $D$  is not contained in the support of  $\Omega$ . We shall derive a contradiction from this assumption.

First, by Lemma 4.3 and Lemma 5.1, we know that

$$\text{mult}_p s > \frac{3m}{2}.$$

By our assumption, we also have  $D \cdot \Omega > 0$ ,  $C \cdot \Omega > 0$  and  $L \cdot \Omega > 0$ . A careful analysis of these three inequalities leads to the following results:

$$\text{mult}_p s \leq 2m, \quad \text{mult}_p \Omega \leq \frac{5m}{6}, \quad \mu > \frac{m}{2}, \quad \nu > \frac{m}{4}.$$

Choose a suitable blow down map  $\pi : \tilde{X} \rightarrow \mathbb{C}P^2$  such that  $\deg \pi(C) = 1 = \deg \pi(L) = \deg \pi(D)$ . Then five out of the six blowing up centers lie on  $\pi(C) \cup \pi(L)$ , with the other one, denoted by  $q$ , lying on  $\pi(D)$ . By Lemma 4.1 and our assumption,  $\pi(\Omega)$  can not contain any line through  $q$ . By computing intersection numbers of  $\pi(\Omega)$  with lines through  $q$ , we can get:

$$\mu \leq m, \quad \nu \leq m.$$

We now further blow up  $\tilde{X}$  at  $p$  with exceptional divisor  $E$ . The blowing up map is  $f : X_1 \rightarrow \tilde{X}$ . Then

$$f^*\left(K_{\tilde{X}} + \frac{2}{3m}Z(s)\right) = K_{X_1} + \frac{2}{3m}\left(\mu \bar{C} + \nu \bar{L} + \bar{\Omega}\right) + \left(\frac{2}{3m}\text{mult}_p s - 1\right)E,$$

where  $\bar{F}$  denotes the strict transform of  $F$  for any divisor  $F$ . Since  $\frac{2}{3m}Z(s)$  is not log terminal at  $p$ , there exists a point  $Q$  on  $E$  such that  $\frac{2}{3m}\left(\mu\bar{C} + \nu\bar{L} + \bar{\Omega}\right) + (\frac{2}{3m}\text{mult}_p s - 1)E$  is not log terminal at  $Q$ .

If  $Q \notin \bar{C}$  and  $Q \notin \bar{L}$ , then

$$\frac{2}{3m}\text{mult}_Q\bar{\Omega} + \left(\frac{2}{3m}\text{mult}_p s - 1\right) \geq 1,$$

hence  $\text{mult}_Q\bar{\Omega} \geq 3m - \text{mult}_p s \geq m$ . But this is impossible in view of the fact  $\text{mult}_p\Omega \leq \frac{5m}{6}$ . Now suppose  $Q \in \bar{L}$ . Then by Fubini's theorem on repeat integration, we have

$$\bar{L} \cdot \left(\frac{2}{3m}\bar{\Omega} + \left(\frac{2}{3m}\text{mult}_p s - 1\right)E\right) \geq 1,$$

which implies that  $\nu \geq m$ . So we have  $\nu = m$  and  $\text{mult}_p\Omega \leq m - \mu < \frac{m}{2}$ . However, by Hölder inequality, this implies  $c_p(s) > \frac{2}{3m}$ .

So there is only one possibility, namely,  $Q \in \bar{C}$ . As in the above discussions, we have

$$\bar{C} \cdot \left(\frac{2}{3m}\bar{\Omega} + \left(\frac{2}{3m}\text{mult}_p s - 1\right)E\right) \geq 1.$$

This implies  $\mu \geq m$ , hence  $\mu = m$ . In this case, we have  $\nu = L \cdot \Omega \geq \text{mult}_p\Omega$  and  $\text{mult}_p\Omega + \nu = \text{mult}_p s - m \leq m$ . Combined with Hölder inequality, these inequalities imply that  $\nu = \text{mult}_p\Omega = \frac{m}{2}$ . This at once leads to a contradiction when  $m$  is odd. When  $m$  is even, this is also impossible by the following lemma:

**Lemma 5.2.** *Let  $f, h$  be germs of holomorphic functions at  $0 \in \mathbb{C}^2$ , with  $f(z_1, z_2) = z_1^{2k}z_2^kh$ , and  $\text{mult}_0h = k$ . Suppose  $z_1 \nmid h$  and  $z_2 \nmid h$  in  $\mathcal{O}_{\mathbb{C}^2, 0}$ , then  $c_0(f) > \frac{1}{3k}$ .*

This lemma can be proved, for example, by induction on the number of blowing ups to resolve the singularity of  $\{f = 0\}$ . The detail is left as an exercise for the reader.  $\square$

### 5.3 Kähler-Einstein metrics on singular cubic surfaces

Even though we have

$$\frac{1}{\alpha_m(X)} + \frac{1}{\alpha_{m,2}(X)} > 3$$

for  $X$ , we can not apply Theorem 1.2 directly to show the existence of Kähler-Einstein metrics on  $X$  due to the presence of singularities. We now explain these difficulties.

Recall that in Tian's proof of the Calabi's conjecture in dimension 2 [26], he first found one surface in the moduli space that admits a Kähler-Einstein metric (this was done in an earlier paper with Yau [31]), then he used the continuity method and solved the Monge-Ampère equations along a regular family of complex surfaces. The key point is the  $C^0$ -estimate, for the higher order estimates differ little from Yau's paper [34]. To prove the  $C^0$ -estimate, there are three main ingredients: the "partial  $C^0$ -estimate", the  $\alpha_m, \alpha_{m,2}$ -invariants estimates and the continuity of rational integrals in dimension 2. The "partial  $C^0$ -estimate" is used to reduce the

$C^0$ -estimate to the uniform estimate of certain rational integrals. Then we can combine the  $\alpha$ -invariants estimates and the continuity of rational integrals to get the uniform  $C^0$ -estimate.

In our case, one may do everything in the category of orbifolds. We can start with a family of cubic surfaces, each has one ordinary double points, together with a family of smooth varying orbifold Kähler metrics. The analysis here is almost identical to that of the smooth case considered in [26], but the problem is that we do not *a priori* have the existence of one Kähler-Einstein orbifold in the moduli space of cubic surfaces with one ordinary double point in view of our computations in Lemma 5.1. So instead, we choose a family of smooth cubic surfaces degenerating to the singular surface  $X$ . We solve the Monge-Ampère equations along this family. However, the  $C^2$ -estimate does not follow easily from the  $C^0$ -estimate in our case, since we do not have a uniform lower bound for the bisectional curvature. Actually, we can expect a “partial  $C^2$ -estimate”, i.e., a  $C^2$ -estimate outside a subvariety. At this point, the technique of [23] and [32] should be helpful. Also, we need a generalized continuity theorem to insure the continuity of rational integrals when the integration domain degenerates to a domain with “mild” singularities.

For general normal cubic surfaces, a theorem of Ding and Tian [9] claims that if the surface has a Kähler-Einstein metric, then it must be semistable. That is to say, the surface can not have singularities other than  $A_1$  and  $A_2$  types. It is expected that the main theorem of [28] still holds in the orbifold case, then we can easily show that a semistable cubic surface with a  $A_2$  singular point can not have Kähler-Einstein metrics except that it has three  $A_2$  singular points. Note that every cubic surface with three  $A_2$  singular points is projectively equivalent to the surface defined by

$$z_0^3 + z_1 z_2 z_3 = 0.$$

It is the quotient of  $\mathbb{C}P^2$  by the cyclic group  $\Gamma_3$ , hence it always has an orbifold Kähler-Einstein metric (See [9]).<sup>5</sup>

A cubic surface with only  $A_1$  singularities (i.e. ordinary double points) can be easily classified as did in [1], [11] and [10]. The number of singular points could be 1, 2, 3 or 4. Any cubic surface with 4 ordinary double points is projectively equivalent to the Segre cubic surface defined by the equation

$$z_0 z_1 z_2 + z_0 z_1 z_3 + z_0 z_2 z_3 + z_1 z_2 z_3 = 0.$$

In [4] Cheltsov proved the existence of orbifold Kähler-Einstein metric on this surface by showing that the  $\alpha_G$ -invariant is bigger than  $2/3$ .

If the cubic surface  $X$  has 2 or 3 ordinary double points, then  $\alpha_m(X) = 1/2$  according to Cheltsov [4]. One can easily show that  $\alpha_{m,2}(X) \leq 1$ . So even if we can overcome all the analytic difficulties mentioned above, we still can not use Tian’s criterion. In this case, one needs a generalized form of Theorem 1.2, also making use of  $\alpha_{m,3}$ . Note that for a cubic surface with 1 or 2 ordinary double points, the  $\alpha_G$ -invariants are equal to the corresponding  $\alpha$ -invariants, due to the existence of certain  $G$ -invariant anticanonical divisors.

All these problems will be discussed in details in another paper.

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<sup>5</sup>In [4], Cheltsov also proved the existence of Kähler-Einstein metric by computing the  $\alpha_G$ -invariant .

## A Appendix

In this appendix, we give a proof of Theorem 2.2. This part is taken from [22].

**Proof of Theorem 2.2:** Since we always have

$$\alpha(X) \leq \inf_m \alpha_m(X),$$

it's clear that we need only to prove:  $\forall c > \alpha(X)$ , we can find some  $m \gg 1$  s.t.  $c > \alpha_m(X)$ .

Now by definition, we can find a sequence of  $\varphi_i \in P(X, g)$ , s.t.

$$\int_X e^{-c\varphi_i} dV_g \rightarrow +\infty, \quad \text{as } i \rightarrow +\infty$$

To make things simple, we assume further that  $\exists \varepsilon_0 > 0$ , s.t.

$$\int_X e^{-(c-\varepsilon_0)\varphi_i} dV_g \rightarrow +\infty, \quad \text{as } i \rightarrow +\infty$$

Set  $\bar{c} = c - \frac{\varepsilon_0}{2}$ . After passing to subsequence, we may assume without loss of generality that  $\varphi_i$  converges to a  $\varphi$  in  $L^1_{loc}$ . Then  $\omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi \geq 0$ .

Define

$$c(\varphi) := \sup\{\alpha > 0 \mid \int_X e^{-\alpha\varphi} dV_g < +\infty\}$$

**Claim 1.**  $\bar{c} > c(\varphi)$

The reason is simple: If  $\bar{c} \leq c(\varphi)$ , then  $\forall \varepsilon > 0$ ,  $\bar{c} - \varepsilon < c(\varphi)$ . Then by Theorem 2.1, we know that

$$\int_X e^{-(\bar{c}-\varepsilon)\varphi_i} dV_g \rightarrow \int_X e^{-(\bar{c}-\varepsilon)\varphi} dV_g < +\infty, \quad \text{as } i \rightarrow +\infty$$

A contradiction.

**Claim 2.**  $\exists k_0 > 0$ , s.t.  $\forall m > k_0$ , there exists a global non-zero holomorphic section  $s$  of  $K^{-m}$ , such that

$$\int_X |s|_{h^m}^2 e^{-(m-k_0)\varphi} dV_g < +\infty$$

We will first prove the theorem assuming this.

$\forall p > 1, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$\begin{aligned} \int_X e^{-(m-k_0)p^{-1}\varphi} dV_g &= \int_X (|s|_{h^m}^2 e^{-(m-k_0)\varphi})^{\frac{1}{p}} (|s|_{h^m}^2)^{\frac{1}{q}-1} dV_g \\ &\leq \left( \int_X |s|_{h^m}^2 e^{-(m-k_0)\varphi} dV_g \right)^{\frac{1}{p}} \left( \int_X |s|_{h^m}^{-2(q-1)} dV_g \right)^{\frac{1}{q}} \end{aligned}$$

If we let  $q - 1 = \frac{\bar{c}}{m}$ , then  $\frac{m-k_0}{p} = \frac{\bar{c}(m-k_0)}{m+\bar{c}} < \bar{c}$ . When  $m$  is big enough,  $\frac{m-k_0}{p}$  is very close to  $\bar{c}$ , hence  $\frac{m-k_0}{p} > c(\varphi)$ . So we have

$$\int_X |s|_{h^m}^{-\frac{2\bar{c}}{m}} dV_g = +\infty$$

That is to say  $c > \bar{c} \geq \alpha_m(X)$ . So we have  $\alpha(X) = \inf_m \alpha_m(X)$ .

It also can be seen from the above proof that for any sequence  $m_k \nearrow +\infty$ ,

$$\alpha(X) = \inf_k \alpha_{m_k}(X).$$

From this, we have

$$\liminf_{m \rightarrow +\infty} \alpha_m(X) = \alpha(X).$$

If

$$\limsup_{m \rightarrow +\infty} \alpha_m(X) = c > \alpha(X),$$

then we have a sequence  $m_k \nearrow +\infty$ , s.t.  $\alpha_{m_k}(X) \rightarrow c$ . Then when  $k$  is sufficiently large, say, larger than some fixed  $k_0$ , we have

$$\alpha_{m_k}(X) > \frac{c + \alpha(X)}{2}.$$

But this is absurd, since we still have

$$\alpha(X) = \inf_{k > k_0} \alpha_{m_k}(X).$$

So we must have  $\alpha(X) = \lim_{m \rightarrow +\infty} \alpha_m(X)$ .  $\square$

**Proof of Claim 2:** The proof is standard. It is motivated by a theorem of Demailly ([6] P110-111). For completeness and the reader's convenience, we include a proof here following [6].

Fix  $x \in X$ , s.t.  $\varphi(x) \neq -\infty$ . Choose a pseudoconvex coordinate neighborhood  $\Omega$  of  $x$ , such that  $K^{-m}$  is trivial over  $\Omega$ . Then by the Ohsawa-Takegoshi theorem<sup>6</sup>, there exists a holomorphic function  $g$  on  $\Omega$  with  $g(x) = 1$  and

$$\int_{\Omega} |g(z)|^2 e^{(m-k_0)\varphi(z)} d\lambda(z) < C_1$$

( $k_0$  to be fixed later). Choose a  $C^\infty$  cut-off function  $\chi : \mathbb{R} \rightarrow [0, 1]$  s.t.  $\chi(t)|_{t \leq 1/2} \equiv 1$ ,  $\chi(t)|_{t \geq 1} \equiv 0$ . Using the trivialization of  $K^{-m}$ , we may view  $\chi(\frac{z-x}{r})g(z)$  as a smooth section of  $K^{-m}$ . We denote it by  $\sigma$  to avoid confusion. Put  $v := \bar{\partial}\sigma$ . We want to solve the equation  $\bar{\partial}u = v$ .

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<sup>6</sup>In fact, Hörmander's  $L^2$  theory is enough. This result is known as the Hörmander-Bombieri-Skoda theorem in the literature.

To make sure the section we shall construct is non-trivial, we introduce a new weight  $e^{-2n\rho_x(z)}$  where  $\rho_x(z) = \chi(\frac{z-x}{r}) \log |z-x|$ . Choose  $k_0 \in \mathbb{N}$  s.t.  $k_0\omega_g + n\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\rho_x \geq 0$ . Then we choose a singular hermitian metric on  $K^{-m}$  to be  $h^m e^{-(m-k_0)\varphi}$ . Since

$$n\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\rho_x + Ric(h) + Ric(h^m e^{-(m-k_0)\varphi}) \geq \omega_g,$$

by Hörmander's  $L^2$  existence theorem, we get a smooth section  $u$  with  $\bar{\partial}u = v$  and

$$\int_X |u|_{h^m}^2 e^{-2n\rho_x-(m-k_0)\varphi} dV_g \leq C_2 \int_{\frac{1}{2} < |z-x| < r} |g|^2 e^{-2n\rho_x-(m-k_0)\varphi} dV_g \leq C_3$$

Since  $\rho_x$  has logarithmic singularity at  $x$ , we have  $u(x) = 0$ . Define  $s = \sigma - u$ , then  $s$  is a non-zero holomorphic section of  $K^{-m}$  with

$$\int_X |s|_{h^m}^2 e^{-(m-k_0)\varphi} dV_g < +\infty.$$

□

It's easy to see that our proof still works in the orbifold case.

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